UNDERSTANDING ADDITIVE AND MULTIPLICATIVE STRUCTURES: THE EFFECT OF NUMBER STRUCTURE AND NATURE OF QUANTITIES ON PRIMARY SCHOOL STUDENTS' PERFORMANCE

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ABSTRACT

The present study explores relationships between additive and multiplicative structures in the context of proportional reasoning. One goal is to examine hypothetical learning trajectories derived from these relations in the context of understanding proportionality. 198 Spanish primary school children were given a test which involved twelve problems with proportional and non-proportional situations in which the relationship between quantities (integer or non-integer) and the nature of quantities (discrete or continuous) were manipulated. Findings revealed the existence of two separate additive and multiplicative structures and that the integer and non-integer relationships between quantities play different roles in these two structures. In additive situations, understanding the integer relationship between quantities was a forerunner to success while in the proportional situations the important thing was to understand non-integer relationships between quantities.

INTRODUCTION

Proportional reasoning implies not only the understanding of the multiplicative relationship that exists between quantities but also the ability to discriminate proportional from non-proportional situations (Christou & Philippou, 2002; Modestou, Elia, & Gagatsis, 2008; Karplus, Pulos, & Stage, 1983). The skills of proportional reasoning are extremely useful in the interpretation of real phenomena because a lot of real-life phenomena follow these proportional rules (Cramer, Post, & Currier (1993)). Furthermore, proportional reasoning is not only relevant in mathematics but also in other sciences like biology, physics, geography, and in many contexts such as monetary changes, change of units, scale drawings, speeds, reductions and enlargements, map reading, etc (Van Dooren, De Bock, Janssens, & Verschaffel, 2008).

Proportional reasoning plays such a critical role in a students' mathematical development that it has been described as a watershed concept, a cornerstone of higher mathematics and the capstone of elementary concepts (Lesh, Post & Behr, 1988). One of the important aspects in the development of proportional reasoning at the end of Primary education is that of the mechanisms which drive the change from additive to multiplicative reasoning. This change is understood as a part of a cognitive development where students' schemas change. These changes in schemas lead students to understand and solve more complex situations (Verschaffel, Greer & Torbeyns, 2006). Vergnaud (1997) introduced the concept of "conceptual field" to characterize these schemas. For example, the conceptual field of the multiplicative structures is designed as a network of interconnected but distinct concepts such us multiplication, division, fractions, ratios, numbers, rational and linear and nonlinear functions.

Some degree of mathematical maturity is required to understand the difference between adding and multiplying and contexts in which each operation is appropriate. One of the difficult tasks for primary school children is to understand the multiplicative nature of the rational numbers. Children who reason additively indiscriminately employ additive transformations, but whether additive reasoning is an invariant stage in the development of proportional reasoning is unclear (Lamon, 2007). So, although the multiplicative structures are based in part on the additive structure, they also have their own specificity that is not reducible to additive structures. The ideas of ratio and proportion exemplify this difference. One question generated in this area concerns the links between the understanding of additive and multiplicative structures (Lamon, 2007).

Steffe (in Lamon, 2007) has articulated a theory concerning the way in which children's formation and use of units progressively develops from early counting through multiplication. The centrality of unit in fraction instruction, especially the role of composite units and the fact that ratios and rates may be viewed as complex types of units suggest that unit building may be an important mechanism in accounting for the development of increasingly sophisticated mathematical ideas. The ability to construct a reference unit and then reinterpret a situation in terms of that unit, (which is called "unitizing", Lamon, 1994) appears critical to the development of increasingly sophisticated mathematical ideas.

In our attempt to study the links between the understanding of multiplicative and additive structures in the development of primary school students, we have linked two aspects of proportional reasoning research. On the one hand, there are studies that have shown students' over-use of proportionality (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; De Bock, Van Dooren, Janssens, & Verschaffel, 2007; Modestou & Gagatsis, 2008; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005;). On the other hand the literature indicates that some variables influence learners' performance in proportional problem-solving (Boyer, Levine, & Huttenlocher, 2008; Harel & Behr, 1989; Jeong, Levine, & Huttenlocher, 2007; Tourniaire & Pulos, 1985; Van Dooren, De Bock, Evers, & Verschaffel, 2009).

Over-use of proportionality: Relations between the additive and multiplicative structure

Learners of different ages tend to apply proportional methods to solve different types of non-proportional situations where one of the quantities is unknown (word missing-value problems) although this is not the appropriate method. This fact shows the difficulty experienced by some learners in distinguishing proportional from non-proportional situations and the tendency to use multiplicative relationships in additive situations (De Bock et al., 2007; Van Dooren et al., 2005; Van Dooren et al., 2008). An example of an additive situation (non-proportional problem with the structure $f(x) = x + b, b \neq 0$) is the following problem: "Sue and Julie are running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many had Sue run?" It was found that that a large number of schoolchildren and even pre-service elementary school teachers responded erroneously to this problem by solving a proportion $9 / x = 3 / 15$ (Cramer et al., 1993; Van Dooren et al., 2005).

Moreover, Modestou & Gagatsis (2007) have provided further evidence that the improper application of proportionality in non-proportional situations, regardless of students' grade and tests' setting, is an indication that linearity is an epistemological obstacle. Therefore, proportional reasoning is a suitable context in which to study the development of multiplicative structure and its relation to additive structure (Kaput & West, 1994; Karplus et al., 1983) and to identify how this relation influences the hypothetical learning trajectories.

Variables affecting proportional reasoning

Research has shown that there are certain factors that affect proportional reasoning and show a complex relation between additive and multiplicative structure. These factors are: the size of the numbers, the existence of integer internal or external ratios (number structure), the existence of continuous or discrete quantities and the familiarity of the context. Some of the results obtained by Van Dooren et al., (2009) indicate that learners are more successful in proportional problems when the external ratio between quantities is an integer (whatever the internal ratio) and the presence of non-integer ratios leads learners to use incorrect additive strategies. The external ratio relates quantities from different magnitudes, while the internal ratio relates quantities from the same magnitude. Furthermore, with regard to non-proportional situations, fewer learners use proportional methods when the relationship between quantities is not an integer. These results indicate that the type of multiplicative relationship between quantities in a non-proportional situation influences the learner's ability to adopt or not to adopt proportional approximations to solve a situation.

On the other hand, the effect of the variable "nature of quantities" in learners' performance when solving proportional problems is still controversial in the literature. Tourniaire and Pulos, (1985) pointed out that students can more easily visualize discrete quantities than continuous ones. So they are more successful in dealing with discrete quantities than with continuous ones in proportional problems. Further studies have suggested the opposite, using numerical comparison problems. For instance, Jeong et al. (2007) showed that children of 6, 8 and 10 years of age use additive strategies when quantities are discrete but use multiplicative relationships with continuous quantities to try to solve proportional problems. Also, Boyer et al., (2008) found that 6 to-9-year- old children had more difficulty when quantities were discrete than when quantities were continuous in proportional problems.

In this study we investigated the interaction effect of these two variables (number structure and nature of quantities) in order to understand better the link between additive and multiplicative structures. For this goal we used proportional situations ($f(x) = ax$) and non-proportional situations ($f(x) = x + b, b \neq 0$) in order to obtain information about how students integrate additive and multiplicative relations in their understanding of the multiplicative conceptual field. More specifically, this study explores de following questions:

- How do the variables of "nature of quantities" (discrete or continuous quantities) and "number structure" (integer or non-integer relationships between quantities) influence learners' performance when solving proportional and additive problems?
- Taking into account the conceptual field of additive and multiplicative structures, how might we explain these influences?
- What are the hypothetical learning trajectories that we can infer from these influences?

METHOD

Participants

The participants were 198 primary school children: 65 in the 4th grade (9-10 years old), 68 in the 5th grade (10-11 years old) and 64 in the 6th grade (11-12 years-old) from two different Spanish schools. The participating schools were located in the same city and the pupils were from mixed socio-economic backgrounds.

The data were collected at the start of the academic year 2008-2009. Contents in the primary school curricula relating to the multiplicative conceptual field are the fraction concept, graphic representations, equivalent fractions and fraction comparisons in 3rd and 4th grades and in grades 5th and 6th, the students are introduced to the computation of percentages of a quantity and to proportional and non-proportional situations.

Instrument and procedure

We used a test with 12 word problems: 4 proportional problems (P), 4 additive problems (A) and 4 buffer problems. We included additive problems *($f(x) = x + b$ with $b \neq 0$)* because the erroneous additive strategy in the proportional problem is a correct strategy in the additive problem and vice versa. Buffer problems were included to avoid learners' discovering the experimental design.

We manipulated the relationship between quantities to obtain integer or non-integer multiplicative relationships. Also, we took into consideration whether the quantities were continuous or discrete. So 2 of the 4 proportional problems and 2 of the 4 additive problems from the test referred to discrete quantities (one where the relationship between quantities is an integer, D-I, and the other where the relationship between quantities is non-integer, D-N). The other 4 referred to continuous quantities (again 2 where the relationship between quantities is an integer, C-I and 2 where the relationship between quantities is a non-integer, C-N). We also limited the size of the numbers (we used numbers with one or two digits), the complexity of the calculation (the outcome is always an integer), the context (always actions) and the position of the unknown quantity (when we read the word problem the unknown quantity is always at the same position). There are examples of the problems in Table 1 and it shows how the experimental variables were manipulated.

To set up the test, we used 8 discrete situations and 8 continuous situations. Then proportional and additive problems were created by manipulating one sentence (for example, "They started together but John plants faster" in the proportional problem and "They plant equally fast but John started earlier" in the additive one). We composed a total of 8 different tests and each test with 4 different orders.

Table 1

Examples of problems considering the number structure (versions I and N) and the nature of quantities (versions D and C)

	I	N
P-D	Rachel and John are planting flowers. They started together but John plants faster. When Rachel has planted 4 flowers, John has planted 12 flowers. If Rachel has planted 20 flowers, how many flowers has John planted?	Rachel and John are planting flowers. They started together but John plants faster. When Rachel has planted 8 flowers, John has planted 12 flowers. If Rachel has planted 20 flowers, how many flowers has John planted?
A-D	Rachel and John are planting flowers. They plant equally fast but John started earlier. When Rachel has planted 4 flowers, John has planted 12 flowers. If Rachel has planted 20 flowers, how many flowers has John planted?	Rachel and John are planting flowers. They plant equally fast but John started earlier. When Rachel has planted 8 flowers, John has planted 12 flowers. If Rachel has planted 20 flowers, how many flowers has John planted?
P-C	Jill and Anthony are painting a fence. They started together but Jill paints slower. When Jill has painted 2 m, Anthony has painted 10 m. If Jill has painted 6 m, how many meters has Anthony painted?	Jill and Anthony are painting a fence. They started together but Jill paints slower. When Jill has painted 20 m, Anthony has painted 50 m. If Jill has painted 30 m, how many meters has Anthony painted?
A-C	Jill and Anthony are painting a fence. They paint equally fast but Jill started later. When Jill has painted 2 m, Anthony has painted 10 m. If Jill has painted 6 m, how many meters has Anthony painted?	Jill and Anthony are painting a fence. They paint equally fast but Jill started later. When Jill has painted 20 m, Anthony has painted 50 m. If Jill has painted 30 m, how many meters has Anthony painted?

The pupils had 50 minutes (i.e. the duration of a regular mathematics lesson) to complete the test. There were no further test instructions except that the children were told that they were allowed to use calculators and were asked to write down the operations they had computed by means of the calculator.

Analysis

Pupils’ responses were analyzed to identify correct (codified as 1) and incorrect (codified as 0) answers. Moreover, strategies that the children used were categorized as proportional (Prop, the use of the multiplicative relationship), additives (Add, the use of the additive relationship) and other (Oth, other incorrect strategies). The analysis of the strategies, however, will not be reported here.

For the analysis of the data an implicative statistical method (Gras, Suzuki, Guillet, & Spagnolo, 2008) was employed using computer software called C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive). The implicative statistical analysis aims at giving a statistical meaning to expressions like: “If we observe the variable *a* in a subject, then in general we observe the variable *b* in the same subject”. The main principle of the implicative analysis is based on the quasi-implication: “If *a* is true then *b* is more or less true”. An implicative diagram has been produced from the application of the analyses on each age group of students. This diagram represents graphically the network of the quasi-implicative relations among the variables considered. In this study the implicative diagrams contain relationships between variables which indicate whether success in a specific problem implies success in another problem. Also responses were statistically analysed by means of a repeated measures logistic regression analysis using the software SPSS. We carried out this analysis to ensure that the differences in learners’ performance were significant.

RESULTS

The results are presented in two parts. In the first part, we present the pupils’ percentages of the correct answers to illustrate their performance in each type of problem. In this way, we study the effect of the variables “nature of quantities” and “number structure”. The second part involves the implicative diagram of the learners’ success level which shows not only the impact of the number structure and nature of quantities but also possible learning trajectories.

The pupils’ performance

Table 2 shows the pupils’ percentages of correct answers to the different problems. In general, they were more successful in the additive problems than in the proportional ones. The statistical analysis indicated that the difference was significant, $\chi^2(1,N=197)=114.530$, $p<0.001$. Furthermore, there is a significant increase in correct answers during grades ($\chi^2(2,N=197)=16.159$, $p<0.001$), except in additive problems with discrete quantities and integer relationship between quantities. While the difference in the pupils’ correct answers was not significant between the 4th and 5th grade (29.23% versus 32.35%), the difference between 5th and 6th grade was significant (32.35% versus 36.13%).

Table 2

Percentages of correct answers to problems

	P-D-I	P-D-N	P-C-I	P-C-N	A-D-I	A-D-N	A-C-I	A-C-N	Total
4th grade	12,31%	1,54%	7,69%	0,00%	56,92%	55,38%	46,15%	53,85%	29,23%
5th grade	23,53%	1,47%	16,18%	1,47%	42,65%	63,24%	45,59%	64,71%	32,35%
6th grade	25,00%	7,81%	20,31%	3,13%	51,56%	62,50%	51,56%	67,19%	36,13%
Total	20,28%	3,61%	14,73%	1,53%	50,38%	60,37%	47,77%	61,91%	32,57%

The pupils were also more successful in proportional problems with integer ratios (20.28% P-D-I and 14.73% P-C-I) than in proportional problems with non-integer ratios (3.61% P-D-N and 1.53% P-C-N). However, in additive problems, they were more successful when the relationships between quantities were non-integer (60.37% A-D-N and 61.91% A-C-N) than when the relationships were integer (50.38% A-D-I and 47.77% A-C-I). The statistical analysis showed a significant “type of problem” × “number structure” interaction effect, $\chi^2(1,N=197)=32.798$, $p<0.001$. These differences, therefore, were significant.

The statistical analysis also showed a significant “type of problem” × “nature of quantities” interaction effect, $\chi^2(1,N=197)=5.657$, $p=0.017$. Table 2 illustrates that learners were more successful in proportional problems with discrete quantities (20.28% P-D-I and 3.61% P-D-N) than in proportional problems with continuous quantities (14.73% P-C-I and 1.53% P-C-N). These differences were significant. In additive problems, although the differences were not significant, the children were also more successful with discrete quantities (50.38% A-D-I and 60.37% A-C-I) than with continuous ones (47.77% A-C-I and 61.91% A-C-N).

Implicative relationships of the pupils’ responses to the tasks

Figure 1 illustrates the implicative diagram of the variables corresponding to 4th- grade-children’s success level in each problem. Two separate “chains” of implicative relations among the variables are formed with respect to the type of problem. Chain 1 involves the learners’ success level with the additive problems and chain 2 is comprised of their success level with the proportional problems. The establishment of these separate implicative chains suggests that no links exist between additive and multiplicative structure, and so children who succeeded in proportional problems did not necessary succeed in additive problems and vice versa. Moreover, the ways in which the different problems are linked in the two implicative chains suggest that success with the proportional or additive problems depended on the “number structure” and “the nature of quantities” involved in the problem. However, the chains do not have a similar structure, so each type of problem depends on the variables in a different way.

The chain related to additive problems indicates that if learners are successful in additive problems with continuous quantities (the relationship between quantities being an integer or non-integer, A-C-N and A-C-I) they are also successful in additive problems with discrete quantities and non-integer relations (A-D-N). Furthermore, the types of situations mentioned above are more likely to lead to success in additive problems with discrete quantities and integer relations (A-D-I). These results seem to underline the role played by the variable nature of quantities in the understanding of the additive situations in primary school children.

With respect to the proportional problems implicative chain, if learners are successful in discrete proportional problems with non-integer ratios (P-D-N) they are also successful in discrete and continuous problems with integer ratios (P-D-I and P-C-I). So it is important in that case to be successful with non-integer ratios to be successful with integer ratios.

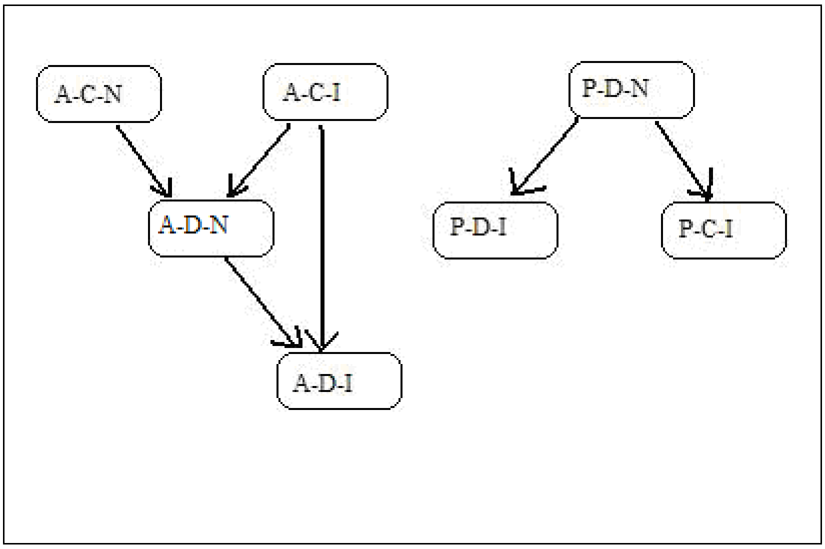

Figure 1. Implicative Diagram for fourth grade pupils

Figure 2 and Figure 3 show the implicative diagrams corresponding to 5th and 6th grade children’s success level respectively. Again two separate “chains” of implicative relations appeared between additive and multiplicative structures. One chain represents pupils’ answers to additive problems (chain 1) and the other chain their answers to proportional problems (chain 2). As we have said above, these two separate chains indicate that there are no relations between the additive and multiplicative structure and therefore it is not possible to establish relations between the pupils’ success in proportional problems and their success in additive problems.

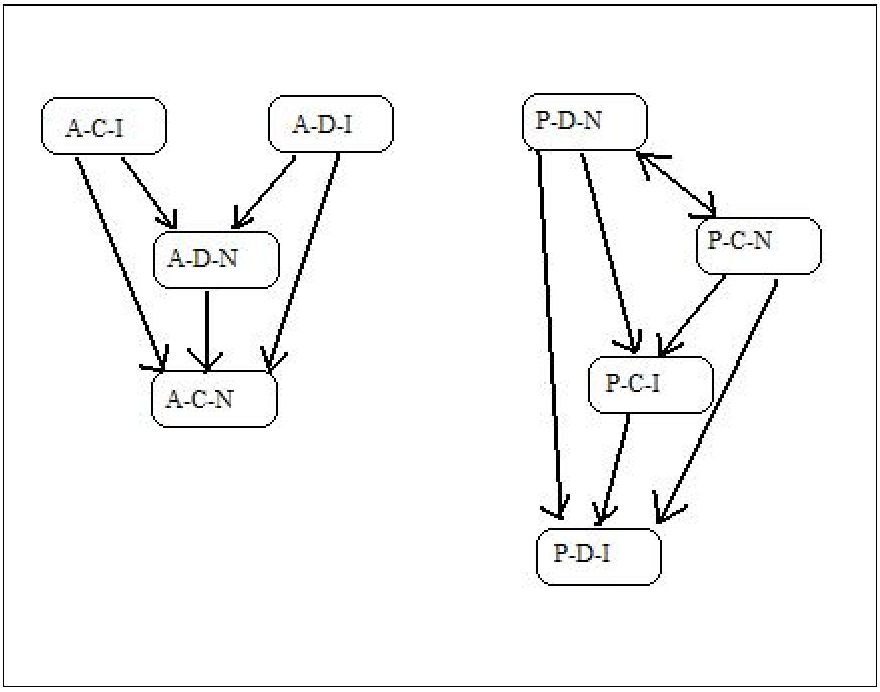

Figure 2. Implicative Diagram for fifth grade pupils

Chain 1, which relates the different additive problems, is the same in 5th and 6th grades. If pupils are successful with integer relationships between numbers (independently of whether they are dealing with discrete or continuous quantities, A-D-I y A-C-I) then, they will be successful with non-integer relationships between numbers (A-D-N y A-C-N). Therefore, the non-integer relations between numbers in the additive problems are showed as more likely to lead to successfully solving this type of problem. Furthermore, if students are successful with discrete quantities and non-integer relationship between numbers (A-D-N) then they will be successful with continuous quantities and non-integer relationship between numbers (A-C-N).

Chain 2, which relates the proportional problems, is different in 5th and 6th grades due to the different role played by the nature of quantities. In 5th grade, if the pupils are successful in proportional problems with non-integer ratios (and with discrete or continuous quantities, P-D-N y P-C-N) then they will be successful with integer ratios (P-D-I y P-C-I). Moreover, if they are successful with discrete quantities and non-integer ratios (P-D-N) they will be successful with continuous quantities and non-integer ratios (P-C-N). And if students are successful with continuous quantities and integer ratios (P-C-I), then they will be successful with discrete quantities and integer ratios (P-D-I).

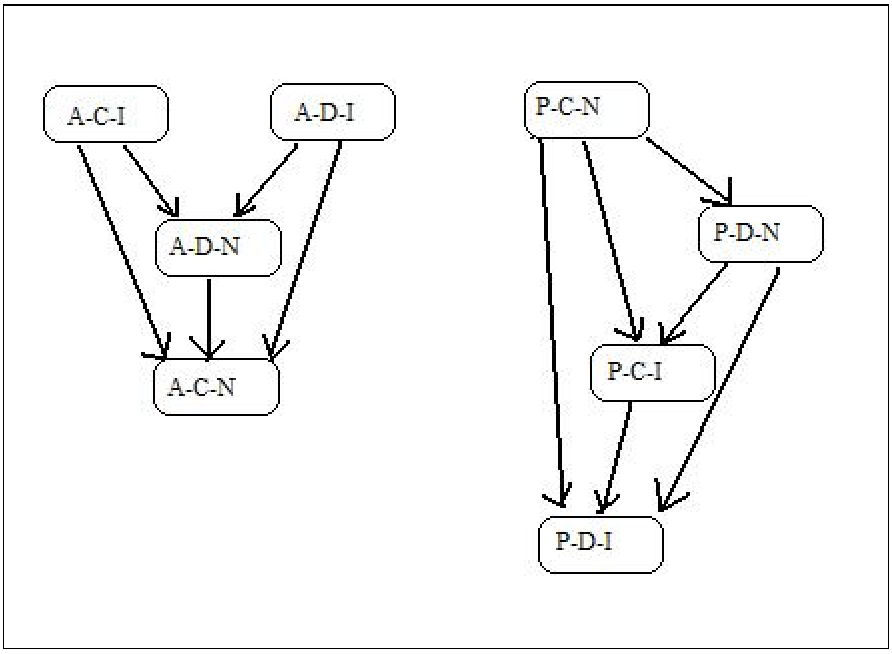

Figure 3. Implicative Diagram for sixth grade pupils

On the other hand, in 6th grade, the relationship between discrete situations with non-integer ratios and continuous quantities with non-integer ratios (P-D-N->P-C-N) disappears entirely. We now find that if learners are successful with non-integer ratios and continuous quantities (P-C-N), then they will be successful with non-integer ratios (P-D-N). Also if they are successful with non-integer ratios (P-D-N and P-C-N), then they will be successful with integer ratios (P-D-I and P-C-I). Moreover, if they are successful with continuous problems and integer ratios (P-C-I), then they also will be successful in discrete problems with integer ratios (P-D-I).

The different implicative relations in the primary school children's success level indicate that the separation between multiplicative and additive structure continues from 4th to 6th grade and that with 5th and 6th grade pupils the nature of quantities has modified the relations established in the particular context of proportional problems.

DISCUSSION

The present study explores relationships between additive and multiplicative structures in the context of proportional reasoning. Specifically, the goal of this study is to investigate how primary school children perform in additive and proportional situations as a way of focusing our attention on how the variables "nature of quantities" and "number structure" affect their understanding of additive and multiplicative structures. Our findings provide information about conceptual operations needed to progressively perform better in the multiplicative conceptual field and more specifically in the context of proportional and non-proportional problems. For this reason, we also examine characteristics of hypothetical learning trajectories derived from the relationships between additive and multiplicative structures in the context of proportional reasoning.

The implicative analysis reveals that additive and multiplicative relationships define two separate structures from 4th through to 6th grade of primary school, so it is not possible to establish relations between pupils' success-levels in additive problems and in proportional problems. These findings seem to indicate that the notion of ratio in the multiplicative structure is generated independently of the additive structures (Kieren, 1994). Other evidence that supports this claim is the large number of students who used proportionality in the additive situation (De Bock et al., 2007; Fernández, Llinares, & Valls, 2008; Modestou & Gagatsis, 2007; Van Dooren et al., 2005; Van Dooren et al., 2008).

Furthermore, the two statistical analyses show the impact of the variables studied ("number structure" and "nature of quantities") on the students' performance in relation to additive and multiplicative structure. With regard to the variable "number structure", the regression analysis indicates that in proportional problems the children perform better when the ratios are integer than when the ratios are non-integer. However, in additive problems, they perform better when the relationship between quantities is non-integer than when it is integer. These results are in agreement with those obtained with the implicative analysis. But also, implicative diagrams seem to indicate that the multiplicative relationship between quantities plays different roles in the two kinds of problems. In additive problems, the integer relationship between quantities is more likely to lead to success, while in proportional problems this role is played by non-integer ratios. Our results regarding the effect of "number structure" obtained with Spanish primary school children replicated those reported by Van Dooren et al. (2009) with Flemish primary school pupils.

On the other hand, in relation to the role played by the nature of quantities in the understanding of the relations between additive and multiplicative structure, in Fernández, Llinares, Van Dooren, De Bock and Verschaffel (in press), the discrete or continuous nature of the quantities did not have a significant effect on secondary school students' performance, but our findings pointed out a possible influence. In this study we dealt only with primary school pupils and this variable does have a significant effect on their performance in proportional problems. Learners perform better in proportional problems with discrete quantities than with continuous ones. These results are in agreement with those of Tourniaire and Pulos, (1985) who in their review of the literature, suggest that learners can more easily visualize discrete quantities than continuous ones. However, later studies have found the contrary (Boyer et al., 2008; Jeong et al., 2007). So this variable is still controversial and requires further research.

Building the meaning of ratio: the cognitive mechanism of unitizing

In proportional problems, primary school students have to construct a reference unit (for example we can represent it by 20:50) and reinterpret the situation in terms of that unit. This process was called by Lamon (1994) “unitizing” and can be considered essential to the ability to differentiate this situation from the additive situation where it is not necessary to construct any new unit. The only requirement there is an additive relation between quantities (in this case 50 – 20).

Learners are more successful in proportional problems with integer ratios than with non-integer ratios, while in additive problems, they are more successful with non-integer ratios than with integer ratios. These findings could be explained by the understanding of the ratio (unit). In the case of integer ratios the unit would be for example 30 : 60. Students could make the division and understand this ratio as the double. But in the case of non-integer ratios, the unit would be for instance 20 : 35. The result of the division is not an integer and it is harder to understand. Students in this type of situation use an additive strategy (additive relationships between quantities).

It is not easy to interpret our results of the variable “nature of quantities” but the fact that children are more successful with discrete quantities than continuous ones could be explained by the idea that the process of addition is associated with situations that entail adding, joining, subtracting, separating and removing- actions with which children are familiar because of their experiences with counting.

Implications for teaching

Our study also suggests some useful guidelines for instruction. Our findings provide information about how primary school children are able to solve additive and proportional problems involving different variables and point out the separation between the processes of building a unit and using it in proportional and additive problems. This situation indicates the necessity for teachers to focus on the difference between these two situations because the notion of ratio in proportional situations doesn’t come from additive structure

Finally, characteristics identified from the implicative analysis enable us to identify hypothetical learning trajectories with implications for teaching. If we consider the implications among problems it should be possible to design teaching situations with different types of problems that progressively raise certain challenges to allow primary school children to gradually overcome obstacles.

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